Topic 1

What is a differential equation?

Ex: $y' = 3y$

To solve this differential equation we want a function y where $y' = 3y$.

Let's try $y = e^{3x}$.

We get $y' = 3e^{3x}$

Notice that here $y' = 3y$.

So, $y = e^{3x}$ solves $y' = 3y$.

Def:

- An equation relating an unknown function and one or more of its derivatives is called a differential equation.

- If a differential equation only has regular derivatives of a single function then its called an ordinary differential equation (ODE).

  If it has partial derivatives then its called a partial differential equation (PDE).

- The order of a differential equation is the order of the highest derivative that occurs in the equation

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Ex: $y' = 3y$

ODE of order 1

Ex: $\underbrace{\frac{dy^2}{d^2x} + \frac{dy}{dx} - 5y = 2}_{y'' + y' - 5y = 2}$

ODE of order 2

<u>Ex:</u> $y'' + 2x^3 y' = \sin(x)$

y is the unknown function

$y = y(x)$ is a function of x

x is a number

ODE of order 2

<u>Ex:</u> (Laplace equation)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Here $u = u(x,y)$ is a function of x and y.

PDE of order 2

<u>Def:</u> An ODE is called <u>linear</u> if it is of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = b(x)$$

these terms only have x's and #'s in them

<u>Ex:</u>

$$2x^2 y''' - 5y' + \frac{1}{x}y = \cos(x)$$

#'s and x's

linear ODE of order 3

Ex:

$$\underbrace{5}y^{(7)}\underbrace{-x}y^{(4)}\underbrace{-}y'+\underbrace{5}=\underbrace{0}$$

#'s & x's

linear ODE of order 7

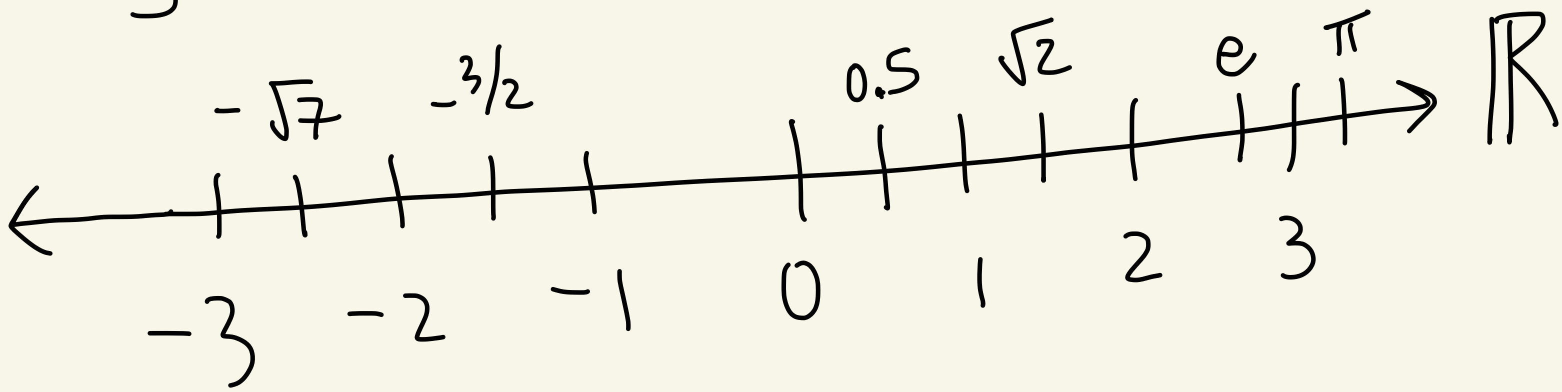
Ex:

$$\underbrace{y^2}_{\text{not \#'s \& x's}}y'\underbrace{-25}y=\underbrace{x}$$

#'s & x's

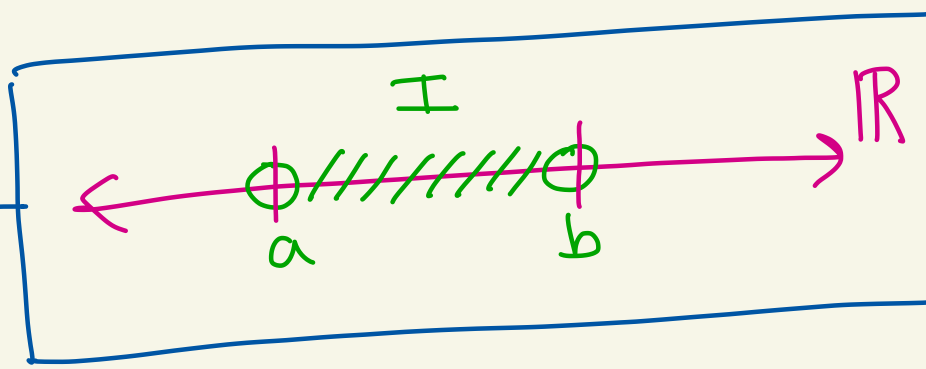
non-linear ODE of order 1

Def: The real number is denoted by $\mathbb{R}$.



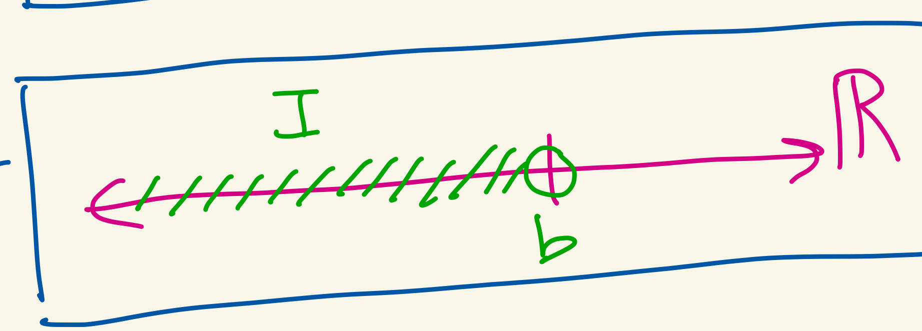
Def: An <u>open interval</u> I is an interval of the form:

$I=(a,b)$



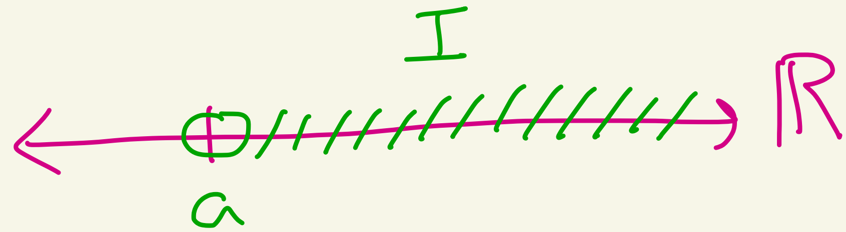
or

$I=(-\infty,b)$



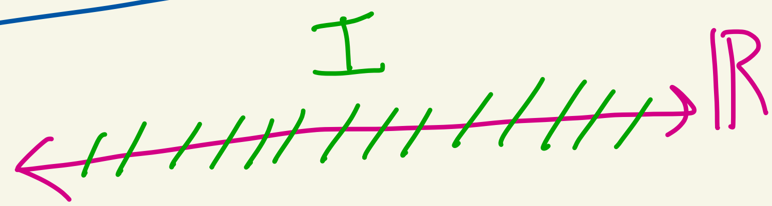
or

$I = (a, \infty)$



or

$I = (-\infty, \infty)$



Def: A function f is a <u>solution</u> to an n-th order ODE on an open interval I if:

1. $f, f', f'', \ldots, f^{(n)}$ exist on I

and

2. when you plug f and it's derivatives into the ODE it solves it for all x in I

In addition, sometimes one is given what

$$f(x_0), f'(x_0), \ldots, f^{(n-1)}(x_0)$$

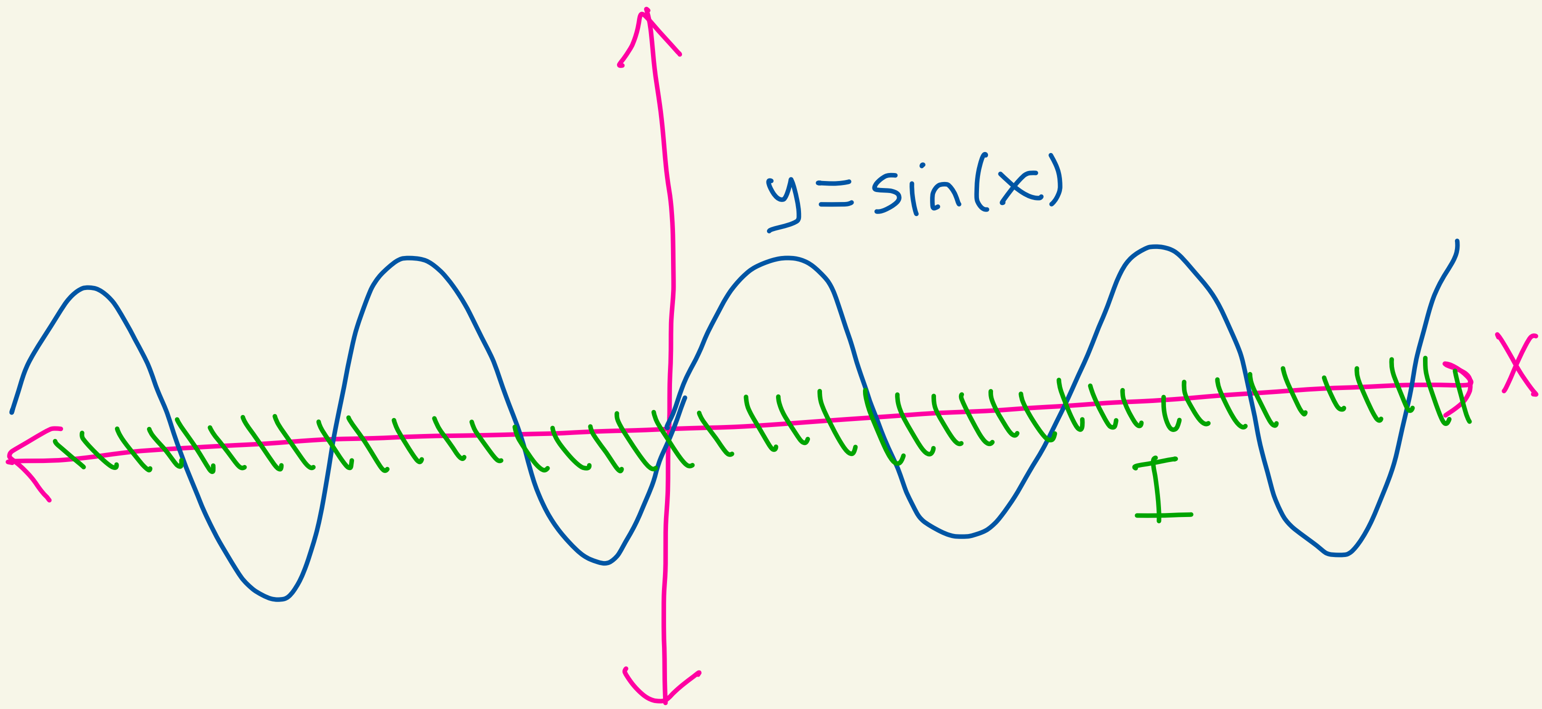
must equal for some x_0 in I. This turns the ODE into an <u>initial value problem</u>.

Ex: Let's find a solution to

$$y'' = -y$$

on $I = (-\infty, \infty)$.

Let $y = \sin(x)$



$y = \sin(x)$ is defined for all $\underbrace{x \text{ in } I}_{-\infty < x < \infty}$.

We have

$$y = \sin(x)$$
$$y' = \cos(x)$$
$$y'' = -\sin(x)$$

So, $y'' = -y$

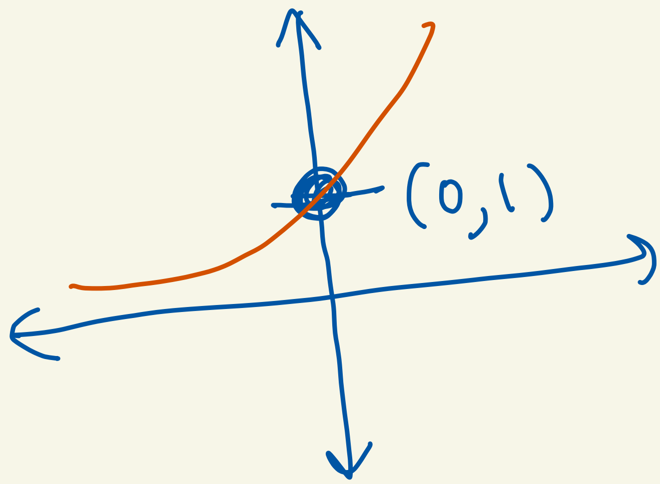
Thus, $y = \sin(x)$ solves $y'' = -y$ on $I = (-\infty, \infty)$.

Ex: Let's find a solution to the initial-valve problem

$$\boxed{\begin{aligned} y' &= y^2 \\ y(0) &= 1 \end{aligned}}$$

nonlinear ODE

condition on solution



Consider $f(x) = \dfrac{1}{1-x}$

Then: $f(x) = (1-x)^{-1}$

$$f'(x) = -(1-x)^{-2} \cdot (-1)$$
$$= (1-x)^{-2} = \frac{1}{(1-x)^2}$$

Then,

$$\underbrace{f'(x)}_{y'} = \frac{1}{(1-x)^2} = \left[\frac{1}{1-x}\right]^2 = \underbrace{[f(x)]^2}_{y^2}$$

So,

$f(x) = \frac{1}{1-x}$ satisfies $y' = y^2$,

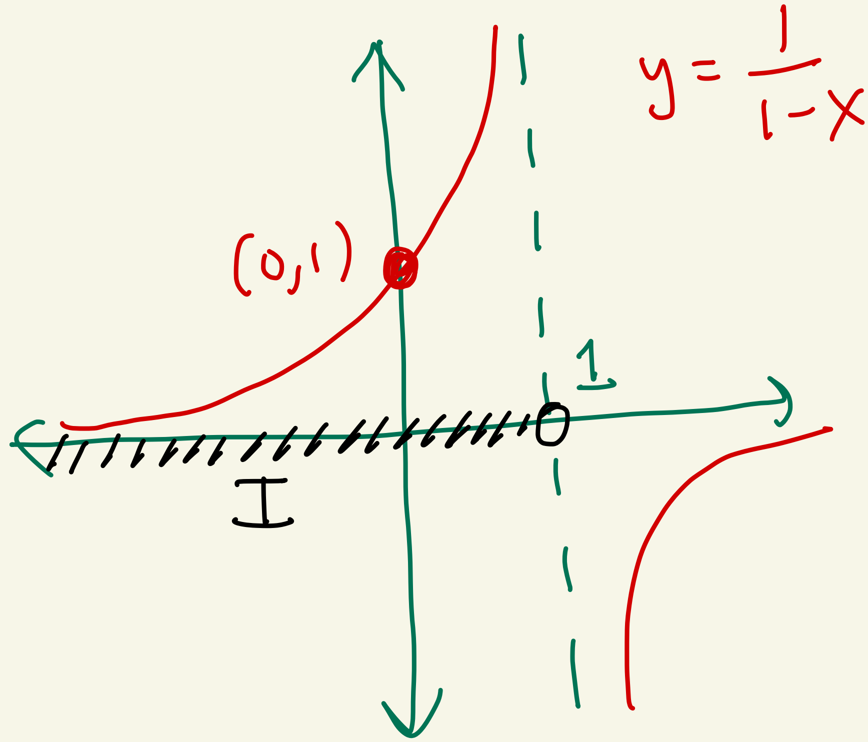
Also,

$$f(0) = \frac{1}{1-0} = 1$$

checking! $y(0)=1$

So, f satisfies the problem

You could say f solves the problem on $I = (-\infty, 1)$



Ex: Given any constants c_1 and c_2 show that

$$f(x) = c_1 e^{2x} + c_2 e^{-2x}$$

satisfies

$$y'' - 4y = 0$$

on $I = (-\infty, \infty)$

Ex: $c_1 = 5$, $c_2 = -3$

$$f(x) = 5e^{2x} - 3e^{-2x}$$

We get:

$$f(x) = c_1 e^{2x} + c_2 e^{-2x}$$

$$f'(x) = 2c_1 e^{2x} - 2c_2 e^{-2x}$$

$$f''(x) = 4c_1 e^{2x} + 4c_2 e^{-2x}$$

these exist for all x that is on $I = (-\infty, \infty)$

Plug in $y'' = f''$ and $y = f$ to get:

$$y'' - 4y = \left(4c_1 e^{2x} + 4c_2 e^{-2x}\right) - 4\left(c_1 e^{2x} + c_2 e^{-2x}\right)$$

$$= 0$$

So, f satisfies $y'' - 4y = 0$ on $I = (-\infty, \infty)$.

END 2(d)

Ex: Find c_1, c_2 where

$$f(x) = c_1 e^{2x} + c_2 e^{-2x}$$

solves the initial-value problem

$$\boxed{\begin{array}{c} y'' - 4y = 0 \\ y'(0) = 0 \\ y(0) = 1 \end{array}}$$

ODE

extra conditions on the solution

We already know from 2(d) that $f(x) = c_1 e^{2x} + c_2 e^{-2x}$ solves $y'' - 4y = 0$.

Let's make it solve the extra conditions,

We have

$$f(x) = c_1 e^{2x} + c_2 e^{-2x}$$
$$f'(x) = 2c_1 e^{2x} - 2c_2 e^{-2x}$$

We need:

$f(0) = 1$
$f'(0) = 0$

$\rightarrow$

$c_1 e^{2(0)} + c_2 e^{-2(0)} = 1$
$2c_1 e^{2(0)} - 2c_2 e^{-2(0)} = 0$

$e^0 = 1$

$c_1 + c_2 = 1$
$2c_1 - 2c_2 = 0$

$\rightarrow$

$c_1 + c_2 = 1$ ①
$c_1 - c_2 = 0$ ②

② gives $c_1 = c_2$.

Plug $c_1 = c_2$ into ① to get
$c_2 + c_2 = 1$. So, $\boxed{c_2 = \frac{1}{2}}$.

Plug back into $c_1 = c_2$ to get $\boxed{c_1 = \frac{1}{2}}$ also.

So,

$$f(x) = c_1 e^{2x} + c_2 e^{-2x}$$
$$= \frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x}$$

is the solution.

$\boxed{\text{END OF 2(e)}}$